$$\text{Ch2}(2.1, 2.2, \overbrace{2.6, 2.7}^{\text{Quantum}}, 2.9, 2.13)$$

(2.1) Consider states $|\psi\rangle = i|\phi_1\rangle + 3i|\phi_2\rangle - |\phi_3\rangle$
and $|\chi\rangle = |\phi_1\rangle - i|\phi_2\rangle + 5i|\phi_3\rangle$

$|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ are orthonormal

a) Calculate $\langle\psi|\psi\rangle$, $\langle\chi|\chi\rangle$, $\langle\psi|\chi\rangle$, $\langle\chi|\psi\rangle$,
and infer $\langle\psi+\chi|\psi+\chi\rangle$.

$$\langle\psi|\psi\rangle = (-i)(i)\langle\phi_1|\phi_1\rangle + (-3i)(3i)\langle\phi_2|\phi_2\rangle + (-)(-)\langle\phi_3|\phi_3\rangle$$

$$= 1 + 9 + 1 = \boxed{11}$$

$$\langle\chi|\chi\rangle = \langle\phi_1|\phi_1\rangle + (i)(-i)\langle\phi_2|\phi_2\rangle + (-5i)(5i)\langle\phi_3|\phi_3\rangle$$

$$= 1 + 1 + 25$$

$$= \boxed{27}$$

$$\langle\psi|\chi\rangle = (1)(-i)\langle\phi_1|\phi_1\rangle + (-i)(-3i)\langle\phi_2|\phi_2\rangle + (+5i)(-)\langle\phi_3|\phi_3\rangle$$

$$= -i - 3 + 5i = \boxed{-3-6i}$$

$$\langle\chi|\psi\rangle = \langle\psi|\chi\rangle^* = \boxed{-3+6i}$$

$$\langle \psi+\chi | \psi+\chi \rangle = 2 \langle \phi_1|\phi_1\rangle + 4\langle\phi_2|\phi_2\rangle + 26\langle\phi_3|\phi_3\rangle$$
$$= \boxed{32}$$

$$\boxed{\begin{aligned} |\psi+\chi\rangle &= (1+i)|\phi_1\rangle + 2i|\phi_2\rangle + (5i-1)|\phi_3\rangle \\ \langle\psi+\chi| &= (1-i)\langle\phi_1| + (-2i)\langle\phi_2| + (-1-5i)\langle\phi_3| \end{aligned}}$$

2.2 Consider two states

$$|\psi_1\rangle = |\phi_1\rangle + 4i|\phi_2\rangle + 5|\phi_3\rangle$$
$$|\psi_2\rangle = b|\phi_1\rangle + 4|\phi_2\rangle - 3i|\phi_3\rangle$$

where $\{|\phi_i\rangle\}$ are orthonormal
b = constant

Find b so that $|\psi_1\rangle$ and $|\psi_2\rangle$ are orthogonal

$$\langle\psi_1|\psi_2\rangle = b\langle\phi_1|\phi_1\rangle + (-4i)4\langle\phi_2|\phi_2\rangle + 5(-3i)\langle\phi_3|\phi_3\rangle$$
$$= b - 16i - 15i$$
$$= b - 31i$$

This must $=0$ for orthogonality, so $\boxed{b = 31i}$

(2.6) Consider a state which is given by

$$|\psi\rangle = \frac{1}{\sqrt{15}}|\phi_1\rangle + \frac{1}{\sqrt{3}}|\phi_2\rangle + \frac{1}{\sqrt{5}}|\phi_3\rangle$$

where $|\phi_n\rangle$ are eigenstates of an operator $\hat{B}$

$$\hat{B}|\phi_n\rangle = (3n^2-1)|\phi_n\rangle \qquad n = 1,2,3$$

a) Find the norm of $|\psi\rangle$

$$\langle\psi|\psi\rangle = \frac{1}{15} + \frac{1}{3} + \frac{1}{5} = \boxed{\frac{3}{5}}$$

b) Find the expectation value of $\hat{B}$ for the state $|\psi\rangle$

$$\frac{\langle\psi|\hat{B}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\langle\psi|\left(\frac{1}{\sqrt{15}}(2)|\phi_1\rangle + \frac{1}{\sqrt{3}}(11)|\phi_2\rangle + \frac{1}{\sqrt{5}}(26)|\phi_3\rangle\right)}{\langle\psi|\psi\rangle}$$

$$= \frac{\frac{2}{15} + \frac{11}{3} + \frac{26}{5}}{3/5} \hat{=} \boxed{15}$$

c) Find expectation value of $\hat{B}^2$ for the state $|\psi\rangle$

$$\hat{B}^2 = \hat{B}\cdot\hat{B}$$

$$\frac{\langle\psi|\hat{B}^2|\psi\rangle}{\langle\psi|\psi\rangle} = \langle\psi|\frac{1}{\sqrt{15}}2^2|\phi_1\rangle + \frac{1}{\sqrt{3}}11^2|\phi_2\rangle + \frac{1}{\sqrt{5}}(26)^2|\phi_3\rangle$$

$$= \frac{\frac{2^2}{15} + \frac{11^2}{3} + \frac{26^2}{25}}{3/5} = \boxed{293}$$

(2.7) Are the following sets of functions linearly independent or linearly dependent?

(a) $4e^x, e^x, 5e^x$

~~Because~~ $5e^x = 4e^x + e^x$, so dependent

(b) $\cos x, e^{ix}, 3\sin x$

$e^{ix} = 1 \cdot \cos x + \frac{i}{3}(3\sin x)$ so dependent

(c) $7, x^2, 9x^4, e^{-x}$

These cannot be written as combinations of one-another, so independent.

(2.9) Are the following sets of vectors linearly independent or dependent?

(a) $(2,-3,0), (0,0,1), (2i, i, -i)$

$$\left.\begin{array}{lll} 2a + 0 = 2ic & \Rightarrow & a = ic \\ -3a + 0 = ic & \Rightarrow & a = -\frac{i}{3}c \\ 0 + b = -ic & \Rightarrow & b = -ic \end{array}\right\}$$ so the only sol'n is $a = b = c = 0$

Thus, the vectors are independent.

(b) $(0,4,0), (i,-3i,i), (2,0,1)$

$$\left.\begin{array}{l} ib + 2c = 0 \\ 4a - 3ib = 0 \\ ib + c = 0 \end{array}\right\}$$ $a = b = c = 0$ is only solution

Vectors are independent

(c) $(i,1,2), (3,i,-1), (-i,3i,5i)$

2.9 (c) $a\cdot(i,1,2)+b\cdot(3,i,-1)+c\cdot(-i,3i,5i)\overset{?}{=}0$

$$i\cdot a+3\cdot b+(-i)\cdot c=0 \xRightarrow{\times i} -a+3ib+c=0$$

$$a+i\cdot b+3i\cdot c=0 \xRightarrow{\times 3} 3a+3ib+9ic=0$$

$$2a+(-b)+5i\cdot c=0 \quad \times i$$

$$\boxed{4a+(9i-1)c=0} \quad \ominus$$

$$\left.\begin{array}{l} 2ai-ib-5c=0 \\ a+ib+3ic=0 \end{array}\right\} \xRightarrow{+} \boxed{a(1+2i)+(-5+3i)c=0}$$

$(1-2i)\times$

$$a(5)+\underbrace{(1-2i)(-5+3i)}_{\substack{(-5+6)+i(3+10) \\ =(1+13i)}}c=0 \Rightarrow \boxed{5a+(1+13i)c=0}$$

$$5(4a+(9i-1)c)=0$$

$$-\quad 4(5a+(1+13i)c)=0$$

$$[5(9i-1)-4(1+13i)]c=0$$

$$[45i-5-4-52i]c=0$$

$$(-97i-9)c=0$$

So $c=0$

Thus $a=0$, $b=0$

$\boxed{\text{Linearly independent}}$

(2.13) In the following expressions, determine whether the result is an operator, bra, ket; then find the Hermitian conjugate.

a) $\underbrace{\langle\phi|\hat{A}|\psi\rangle}_{\text{complex }\#}\underbrace{\langle\psi|}_{\text{bra}} \Rightarrow \boxed{\text{bra}}$

$$\left(\langle\phi|\hat{A}|\psi\rangle\langle\psi|\right)^\dagger = \boxed{\langle\phi|A|\psi\rangle^* \; |\psi\rangle}$$

b) $\hat{A}|\psi\rangle\langle\phi| \rightarrow \boxed{\text{operator}}$

$$(\hat{A}|\psi\rangle\langle\phi|)^\dagger = \boxed{|\phi\rangle\langle\psi|\hat{A}^\dagger}$$

c) $\underbrace{\langle\phi|\hat{A}|\psi\rangle}_{\text{scalar}} \; \underbrace{|\psi\rangle\langle\phi|}_{\text{operator}}\underbrace{\hat{A}}_{\text{operator}} = \text{operator}$

$$\left(\langle\phi|\hat{A}|\psi\rangle \; |\psi\rangle\langle\phi|\hat{A}\right)^\dagger = \boxed{\langle\phi|\hat{A}|\psi\rangle^* \; \hat{A}^\dagger|\phi\rangle\langle\psi|}$$

d) $\left(\langle\psi|\hat{A}|\phi\rangle \underbrace{|\phi\rangle}_{\text{ket}} + i\underbrace{\hat{A}|\psi\rangle}_{\text{ket}}\right) = \text{ket}$

$$(\qquad)^\dagger = \boxed{\begin{array}{l}\langle\psi|\hat{A}|\phi\rangle^*\langle\phi| \\ \quad + (-i)\langle\psi|\hat{A}^\dagger\end{array}}$$