Ammonia maser example

QM 2.11, 2.12, 2.14, 2.16, 2.19(a)(b), 2.22, 2.23, 2.49

2.11 Show: if $\hat{A}$ is a projection operator, $1-\hat{A}$ is too.

Def: Projection operator

$$P^{\dagger} = P$$

$$P^2 = P$$

Therefore $\hat{A}^{\dagger} = A$ and $\hat{A}^2 = A$

$$(\mathbb{1}-\hat{A})^{\dagger} = \mathbb{1}^{\dagger} - \hat{A}^{\dagger} = \mathbb{1} - \hat{A}$$

$$\begin{aligned}(\mathbb{1}-\hat{A})^2 &= (\mathbb{1}-\hat{A})(\mathbb{1}-\hat{A}) \\ &= (\mathbb{1}-\hat{A})\mathbb{1} + (\mathbb{1}-\hat{A})(-\hat{A}) \\ &= (\mathbb{1}-\hat{A}) + \mathbb{1}(-\hat{A}) + (-\hat{A})(-\hat{A}) \\ &= \mathbb{1} - \hat{A} - \hat{A} + \hat{A}^2 \\ &= \mathbb{1} - \hat{A} - \cancel{\hat{A}} + \cancel{\hat{A}} \\ &= \mathbb{1} - \hat{A}\end{aligned}$$

Therefore

$1-\hat{A}$ is also a projection operator

(2.12) show $\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$ is a projection operator regardless whether $|\psi\rangle$ is normalized.

$$\left(\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}\right)^\dagger = \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$$

since $|\psi\rangle^\dagger = \langle\psi|$

and $\langle\phi|\psi\rangle^\dagger = \langle\psi|\phi\rangle$, so the first property is satisfied.

scalar

$$\left(\frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}\right)^2 = \frac{|\psi\rangle\langle\psi|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle^2}$$

$$= \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} \cancel{\frac{\langle\psi|\psi\rangle}{\langle\psi|\psi\rangle}}$$

$$= \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle}$$

(2.14) Consider $\hat{A}$ such that:

$$\hat{A}\,|\phi_1\rangle = |\phi_1\rangle$$
$$\hat{A}\,|\phi_2\rangle = -|\phi_2\rangle$$

$|\phi_1\rangle, |\phi_2\rangle$ orthonormal

a) Do the states form a basis? Note $|\phi_1\rangle, |\phi_2\rangle$ are eigenvectors of $\hat{A}$. One approach is to check if $\hat{A}$ is Hermitian. If so, Theorem 2.2 guarantees that its eigenvectors form a complete basis. So:

$$\hat{A} = \begin{array}{c|cc} & |\phi_1\rangle & |\phi_2\rangle \\ \hline \langle\phi_1| & 1 & 0 \\ \langle\phi_2| & 0 & -1 \end{array}$$

This is Hermitian, since $A = A^\dagger$,

so $\boxed{\text{yes } \{|\phi\rangle\} \text{ is a basis by Thm 2.2.}}$

b) Define $\hat{B} = |\phi_1\rangle\langle\phi_2|$. Is $\hat{B}$ Hermitian?

$$B^\dagger = |\phi_2\rangle\langle\phi_1| \neq B \quad \text{so, } \boxed{\text{no}}$$

b)' Show $\hat{B}^2 = 0$

$$\hat{B}^2 = \hat{B}\cdot\hat{B} = |\phi_1\rangle\langle\phi_2| \cdot |\phi_1\rangle\langle\phi_2| = |\phi_1\rangle\underbrace{\langle\phi_2|\phi_1\rangle}_{0}\langle\phi_2|$$

$$= \boxed{0}$$

c) Show $BB^\dagger$ and $B^\dagger B$ are projection operators.

A projection operator must satisfy $P^\dagger = P$ and $P^2 = P$

$$\boxed{BB^\dagger = |\phi_1\rangle\langle\phi_2| \cdot |\phi_2\rangle\langle\phi_1| = |\phi_1\rangle\langle\phi_1|}$$

This is the form of a projection operator, as shown in class.

Similarly $B^\dagger B = |\phi_2\rangle\langle\phi_1|\phi_1\rangle\langle\phi_2| = \boxed{|\phi_2\rangle\langle\phi_2|}$

$\boxed{\text{Yes}}$

d) Show $BB^\dagger - B^\dagger B$ is unitary

We know from part (c) that

$$Q \equiv BB^\dagger - B^\dagger B = |\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|$$

Also, a unitary operator is defined as

$$\boxed{U^\dagger = U^{-1}} \quad \text{or} \quad \boxed{UU^\dagger = \mathbb{1}}$$

$$\boxed{\text{Try } QQ^\dagger \stackrel{?}{=} \mathbb{1}}$$

$$Q^\dagger = Q$$

$$QQ^\dagger = \big(|\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|\big)\big(|\phi_1\rangle\langle\phi_1| - |\phi_2\rangle\langle\phi_2|\big)$$

$$= |\phi_1\rangle\langle\phi_1|\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|\phi_2\rangle\langle\phi_2| \underbrace{+0 \quad +0}_{\text{from cross terms}}$$

$$= |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|$$

$$= \sum_i |\phi\rangle\langle\phi|$$

Hermitian operator

$$\boxed{QQ^\dagger = \mathbb{1}} \qquad \boxed{\text{So, yes, unitary}}$$

Note: You could also do this by writing the matrices

$$B = |\phi_1\rangle\langle\phi_2| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}$$

$$B^\dagger = |\phi_2\rangle\langle\phi_1| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}$$

$$BB^\dagger = |\phi_1\rangle\langle\phi_1| = \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1 & 0\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}$$

$$B^\dagger B = |\phi_2\rangle\langle\phi_2| = \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}$$

Then $Q = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ and $QQ^\dagger = ($

e) Consider $\hat{C} = BB^\dagger + B^\dagger B$. Show that $\hat{C}|\phi_1\rangle = |\phi_1\rangle$

$\hat{C}|\phi_2\rangle = |\phi_2\rangle$

$$\hat{C} = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|$$
$$= \mathbb{1} \text{ by Thm. 2.2}$$

$$\boxed{\text{Thus } \hat{C}|\phi_1\rangle = |\phi_1\rangle \quad \hat{C}|\phi_2\rangle = |\phi_2\rangle}$$

2.16 a) Verify $U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is unitary

By definition, a unitary operator satisfies

$$U^\dagger = U^{-1} \text{ or } U^\dagger U = \mathbb{1}$$

$$U^\dagger U \stackrel{?}{=} \mathbb{1} \Rightarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{pmatrix} \cdot \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos^2+\sin^2\theta & 0 \\ 0 & \cos^2+\sin 2 \end{pmatrix}$$

$$= \mathbb{1}$$

$$\boxed{\text{So, yes, } U \text{ is unitary}}$$

b) Find the eigenvalues and eigenvectors

$$0 = \begin{vmatrix} \cos\theta - \lambda & -\sin\theta \\ \sin\theta & \cos\theta - \lambda \end{vmatrix} \Rightarrow (\cos\theta - \lambda)^2 + \sin^2\theta = 0$$

$$\cos^2\theta + \lambda^2 - 2\cos\theta\lambda + \sin^2\theta = 0$$

$$1 - 2\cos\theta\lambda + \lambda^2 = 0 \Rightarrow \lambda = \frac{2\cos\theta \pm \sqrt{(2\cos\theta)^2 - 4}}{2}$$

$$= \cos\theta \pm \sqrt{\cos^2\theta - 1}$$

$$= \cos\theta \pm i\sin\theta$$

$$\boxed{\lambda = e^{\pm i\theta}}$$

Find the eigenvectors

$\lambda = e^{+i\theta}$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = e^{i\theta}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\cancel{\cos\theta\, x} - \cancel{\sin\theta}\, y = e^{i\theta} x = \cancel{\cos\theta\, x} + i \cancel{\sin\theta}\, x$$

$$-y = ix$$

Set $x=1$, then $y=-i$

$$\boxed{e^{i\theta} \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}}$$

$\lambda = e^{-i\theta}$

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = e^{-i\theta}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\cancel{\cos\theta\, x} - \cancel{\sin\theta}\cdot y = \cancel{\cos\theta\, x} - i \cancel{\sin\theta}\, x$$

$$-y = -ix$$

Set $x=1$, then $y=i$

$$\boxed{e^{-i\theta} \longleftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}}$$

2.19 (a) $\hat{A} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

$$\begin{vmatrix} -a & i \\ -i & -a \end{vmatrix} = 0 = a^2 - 1$$

$$a = \pm 1$$

$$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = +\begin{pmatrix} x \\ y \end{pmatrix}$$

$$iy = x$$

$x = 1,\ y = -i \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$ eigenvector of $+1$

$x = 1,\ y = +i \Rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ +i \end{pmatrix}$ eigenvector of -1

(b) Is $A^\dagger = A$?

$\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}^\dagger = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, so yes.

Since **A** is Hermitian, its eigenvectors form a complete basis set, and orthonormal by <u>Thm 2.2</u>.

2.22

$$\hat{A} = \begin{pmatrix} 0 & 0 & -1+i \\ 0 & 3 & 0 \\ -1-i & 0 & 0 \end{pmatrix}$$

a) Find eigenvalues and eigenvectors

$$\det(\hat{A} - a\hat{I}) = \begin{vmatrix} -a & 0 & -1+i \\ 0 & 3-a & 0 \\ -1-i & 0 & -a \end{vmatrix} = 0$$

Expand by minors in terms of $(3-a)$

$$\det\hat{A} = (3-a)(-)\begin{vmatrix} -a & -1+i \\ -1-i & -a \end{vmatrix} = 0$$

$$(3-a) = 0 \ \text{OR} \ \underbrace{a^2 - (-)(1+i)(1-i) = 0}_{a^2+2=0}$$

Therefore $a = 3$ OR $a = \pm\sqrt{2}$

Eigenvectors

$a=3$	$a=+\sqrt{2}$	$a=-\sqrt{2}$
$\hat{A}\lvert\psi\rangle = a\lvert\psi\rangle$		
$(-1+i)z = 3x \rightarrow z = \frac{3}{-1+i}x = \frac{3(-1-i)}{2}x$	$(-1+i)z = \sqrt{2}x$	$(-1+i)z = -\sqrt{2}x$
$3y = 3y$	$3y = \sqrt{2}y$	$3y = -\sqrt{2}y$
$(-1-i)x = 3z \rightarrow z = \frac{-1-i}{3}x$	$(-1-i)x = \sqrt{2}z$	$(-1-i)x = -\sqrt{2}z$
true for any y, so set $y=1$; true only if $x=z=0$	True if $y=0$; Pick $x=1$ then $z = -\frac{1}{\sqrt{2}}(1+i)$	True if $y=0$; Pick $x=1$ then $z = \frac{1}{\sqrt{2}}(1+i)$
So $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	So $\begin{pmatrix} 1 \\ 0 \\ -\frac{1}{\sqrt{2}}(1+i) \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ +\frac{1}{\sqrt{2}}(1+i) \end{pmatrix}$
Normalized $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -\frac{1}{\sqrt{2}}(1+i) \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ \frac{1}{\sqrt{2}}(1+i) \end{pmatrix}$

b) Show $\lvert a_1\rangle\langle a_1\rvert + \lvert a_2\rangle\langle a_2\rvert + \lvert a_3\rangle\langle a_3\rvert = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{\sqrt{2}}(1+i) \end{pmatrix}\begin{pmatrix} 1 & 0 & -\frac{1}{\sqrt{2}}(1-i) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ \frac{1}{\sqrt{2}}(1+i) \end{pmatrix}\begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}}(1-i) \end{pmatrix}$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & -\frac{1}{\sqrt{2}}(1-i) \\ 0 & 0 & 0 \\ -\frac{1}{\sqrt{2}}(1+i) & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}}(1-i) \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{2}}(1+i) & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

Check the ~~[illegible]~~ ~~[illegible]~~ eigenvectors to see if orthogonal

$$(0\ \ 1\ \ 0)\begin{pmatrix}1\\0\\\pm\frac{1}{2}(1+i)\end{pmatrix} = 0$$

$$\left(1,\ 0,\ -\frac{1}{\sqrt{2}}(1-i)\right)\begin{pmatrix}1\\0\\+\frac{1}{\sqrt{2}}(1+i)\end{pmatrix} = 1+0-\frac{1}{2}(2) = 0$$

c) Find $P = |a_1\rangle\langle a_1|$

$$\begin{pmatrix}0\\1\\0\end{pmatrix}(0\ \ 1\ \ 0) = \begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix}$$

Yes, by earlier problem

2.23 In a 3D vector space consider operator

$$\hat{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

on an orthonormal basis $|1\rangle, |2\rangle, |3\rangle$

a) Is $\hat{A}$ Hermitian?

Definition of Hermitian operator is

$$\hat{A}^{\dagger} = A = (A^{T})^{*}$$

$$\boxed{\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \overset{?}{=} \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{Yes!}}$$

$\boxed{\text{It is Hermitian.}}$

a') Calculate its eigenvalues and normalized eigen vectors.

Apply $\det(\hat{A} - \lambda \mathbb{I}) = 0$

$\boxed{\text{Expand}}$

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = 0$$

Here it is useful to "expand by minors." Identify the row or column wi the most zeroes and expand.

$$0 + -(1+\lambda) \cdot \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} + 0 = 0$$

$\boxed{\text{minor determinant}}$

$$-(\lambda+1) \cdot (\lambda^2 - 1) = 0$$

$$(\lambda+1)(\lambda+1)(\lambda-1) = 0$$

$\boxed{\text{Eiginvalues}}$

degenerate eigen values $\boxed{\lambda = -1}$ $\boxed{\lambda = +1}$

a') (continued) Now find the eigenvectors. Start w/ the non-degenerate one

$$\hat{A}|\phi\rangle = \lambda|\phi\rangle$$

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = +\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} z = x \\ -y = +y \\ x = z \end{matrix} \Rightarrow y = 0$$

So we must have $y=0$, $x=z$, so set $x=1 \Rightarrow |\phi\rangle = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

Normalizing: $\langle\phi|\phi\rangle|a|^2 = |a|^2 \cdot (2) = 1 \Rightarrow |a| = \frac{1}{\sqrt{2}}$

$$\boxed{\text{So } |\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} \longleftrightarrow \boxed{\lambda = +1}$$

For the degenerate eigenvalue $\lambda = -1$:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -1\begin{pmatrix} x \\ y \\ z \end{pmatrix} \Rightarrow \begin{matrix} z = -x \\ -y = -y \\ z = -x \end{matrix}$$

So y may be anything, $z = -x$. Choose $y = 0$, $x = 1$

So $|\phi_2\rangle = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Normalizing $\boxed{|\phi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \longleftrightarrow \boxed{\lambda = -1}$

The last eigenvector may be chosen orthogonal to the other two: $\boxed{|\phi_3\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} \longleftrightarrow \boxed{\lambda = -1}$

degenerate

a'') Verify the eigenvectors are orthogonal:

~~single~~ nondegenerate

$$\langle\phi_2|\phi_1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$= \frac{1}{2}(1-1) = \boxed{0}$$

Yes!

b) Calculate the matrices representing the projection operators for the two nondegenerate eigenvectors

Projection operators are of the form $|\psi\rangle\langle\psi|$, so

$$|\phi_1\rangle\langle\phi_1| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & 1\end{pmatrix} = \boxed{\frac{1}{2}\begin{pmatrix}1 & 0 & 1\\0 & 0 & 0\\1 & 0 & 1\end{pmatrix}}$$

$$|\phi_2\rangle\langle\phi_2| = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}\cdot\frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0 & -1\end{pmatrix} = \boxed{\frac{1}{2}\begin{pmatrix}1 & 0 & -1\\0 & 0 & 0\\-1 & 0 & 1\end{pmatrix}}$$

2.49 Hamiltonian for a two state system

$$\hat{H} = \varepsilon \begin{pmatrix} 1 & -i \\ +i & -1 \end{pmatrix}$$

a) $\hat{H}^{\dagger} = \varepsilon \begin{pmatrix} 1 & -i \\ +i & -1 \end{pmatrix} = \hat{H}$ so yes, its Hermitian

$\mathrm{Tr}\,\hat{H} = 0$

b)
$$\begin{vmatrix} 1-a & -i \\ i & -1-a \end{vmatrix} = 0$$

$$-(1-a)(1+a) - 1 = 0$$

$$a^2 - 1 - 1 = 0$$

$$a = \pm\sqrt{2}$$

Multiply by ε, so

$$a = \pm\sqrt{2}\,\varepsilon$$

$$\hat{H}|\psi\rangle = \sqrt{2}\,\varepsilon|\psi\rangle$$

$x - iy = \sqrt{2}\,x \rightarrow -iy = (\sqrt{2}-1)x \rightarrow \frac{-i}{\sqrt{2}-1}y = x \rightarrow -i(\sqrt{2}+1)y = x$

$ix - y = \sqrt{2}\,y \rightarrow (1+\sqrt{2})y = ix \rightarrow -i(1+\sqrt{2})y = x$

Set $x = 1$

$y = (\sqrt{2}-1)i$

Normalize $\left(1 + (\sqrt{2}-1)^2\right)c^2 = 1$

$$c = \frac{1}{\sqrt{1+(\sqrt{2}-1)^2}}$$

$$\frac{1}{\sqrt{1+(\sqrt{2}-1)^2}}\begin{pmatrix} 1 \\ (\sqrt{2}-1)i \end{pmatrix}$$

The eigenvector for $-\sqrt{2}$ is

$$\frac{1}{\sqrt{1+(\sqrt{2}-1)^2}}\begin{pmatrix} 1 \\ -(\sqrt{2}-1)i \end{pmatrix}$$