Quantum Mechanics

HW3 Ch3(3.8, 3.14, 3.16, 3.17, 3.18, 3.21, 3.23, 3.27)
Ch2 (Prove theorem 2.1)

3.8 The components of the initial state $|\psi_i\rangle$ of a quantum system are given in a complete and orthonormal basis of three states $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ as

$$\langle\phi_1|\psi_i\rangle = \frac{i}{\sqrt{3}} \qquad \langle\phi_2|\psi_i\rangle = \sqrt{\frac{2}{3}} \qquad \langle\phi_3|\psi_i\rangle = 0$$

Calculate the probability of finding the system in state $|\psi_f\rangle$ given by

$$\langle\phi_1|\psi_f\rangle = \frac{1+i}{\sqrt{3}} \qquad \langle\phi_2|\psi_f\rangle = \frac{1}{\sqrt{6}} \qquad \langle\phi_3|\psi_f\rangle = \frac{1}{\sqrt{6}}$$

- According to the postulates of QM, probability of finding system in state $|\psi_f\rangle$ after initially being in state $|\psi_i\rangle$ is

$$P = \frac{|\langle\psi_f|\psi_i\rangle|^2}{\underbrace{\langle\psi_i|\psi_i\rangle}_{1}} = \left| \left(\frac{1-i}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \begin{pmatrix} i/\sqrt{3} \\ \sqrt{2/3} \\ 0 \end{pmatrix} \right|^2$$

Remember to conjugate!

$$= \left| \frac{i+1}{3} + \sqrt{\frac{2}{18}} + 0 \right|^2$$

$$= \left| \sqrt{\frac{2}{18}} + \frac{1}{3} + \frac{i}{3} \right|^2 = \left| \frac{1}{3} + \frac{1}{3} + \frac{i}{3} \right|^2 = \left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 = \boxed{\frac{5}{9}}$$

(3.14) The initial state of a system is given in terms of four orthonormal energy eigenfunctions $|\phi_1\rangle \dots |\phi_4\rangle$ as

$$|\psi_0\rangle = |\psi(t=0)\rangle = \frac{1}{\sqrt{3}}|\phi_1\rangle + \frac{1}{2}|\phi_2\rangle + \frac{1}{\sqrt{6}}|\phi_3\rangle + \frac{1}{2}|\phi_4\rangle$$

a) If the $|\phi_i\rangle$ are eigenvectors of the Hamiltonian $\mathbb{H}$ with energies E_1, E_2, E_3, E_4 find $|\psi(t)\rangle$ at any later time.

Normalize $|\psi(0)\rangle$:

$$\langle\psi(0)|\psi(0)\rangle = \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{4} = \frac{1}{2} + \frac{3}{6} = 1$$

Good! Already normalized.

Recall that $|\psi(t)\rangle = \sum_i c(0)|\phi_i\rangle e^{-i\omega_i t}$

$$\omega_i = \frac{E_i}{\hbar}$$

$$\boxed{\begin{aligned}\text{Therefore } |\psi(t)\rangle = &\frac{1}{\sqrt{3}}\exp[-i\omega_1 t]|\phi_1\rangle \\ &+ \frac{1}{2}\exp[-i\omega_2 t]|\phi_2\rangle \\ &+ \frac{1}{\sqrt{6}}\exp[-i\omega_3 t]|\phi_3\rangle \\ &+ \frac{1}{2}\exp[-i\omega_4 t]|\phi_4\rangle\end{aligned}}$$

b) What are the possible results of measuring the energy and with what probability?

The possible measurements of $\mathbb{H}$ are the eigenvalues $E_1, \dots, E_4$

$\updownarrow$ $\qquad$ $\updownarrow$

Eigenstate $|\phi_1\rangle \cdots |\phi_4\rangle$

The probability associated with each state is

$$\frac{|\langle\phi_i|\psi\rangle|^2}{\langle\psi|\psi\rangle} \Rightarrow$$

$\langle\psi|\psi\rangle$ 1 (since normalized)

For...	... probability is ...
$|\phi_1\rangle$	$|\langle\phi_1|\psi\rangle|^2 = \left|\frac{1}{\sqrt{3}} e^{-i\omega_1 t}\right|^2 = \boxed{\frac{1}{3}}$
$|\phi_2\rangle$	$|\langle\phi_2|\psi\rangle|^2 = \left|\frac{1}{2} e^{-i\omega_2 t}\right|^2 = \boxed{\frac{1}{4}}$
$|\phi_3\rangle$	$|\langle\phi_3|\psi\rangle|^2 = \left|\frac{1}{\sqrt{6}} e^{-i\omega_3 t}\right|^2 = \boxed{1/6}$
$|\phi_4\rangle$	$|\langle\phi_4|\psi\rangle|^2 = \left|\frac{1}{2} e^{-i\omega_4 t}\right|^2 = \boxed{1/4}$

c) Find the expectation value $\boxed{\text{of } \mathbb{H}}$ at $t = 0s$ and $t = 10s$

$$\langle\mathbb{H}\rangle = \left.\frac{\langle\psi|\mathbb{H}|\psi\rangle}{\langle\psi|\psi\rangle}\right|_{t=0} = \left(\frac{1}{\sqrt{3}}, \frac{1}{2}, \frac{1}{\sqrt{6}}, \frac{1}{2}\right)\begin{pmatrix} E_1 & & & 0 \\ & E_2 & & \\ & & E_3 & \\ 0 & & & E_4 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{2} \\ 1/\sqrt{6} \\ 1/2 \end{pmatrix}$$

basis of $\mathbb{H}$

$$= \left(\frac{1}{\sqrt{3}}, \frac{1}{2}, \frac{1}{\sqrt{6}}, \frac{1}{2}\right)\begin{pmatrix} E_1/\sqrt{3} \\ E_2/2 \\ E_3/\sqrt{6} \\ E_4/2 \end{pmatrix}$$

$$\boxed{\langle\mathbb{H}\rangle = \frac{1}{3}E_1 + \frac{1}{4}E_2 + \frac{1}{6}E_3 + \frac{1}{4}E_4}$$

At a later time t:

$$\langle \mathbb{H} \rangle\big|_t = \langle \psi(t) | \, \mathbb{H} \, | \psi(t) \rangle$$

$$= \left(\frac{1}{\sqrt{3}} e^{+i\omega_1 t}, \frac{1}{2} e^{+i\omega_2 t}, \frac{1}{\sqrt{6}} e^{+i\omega_3 t}, \frac{1}{2} e^{+i\omega_4 t} \right) \cdot \begin{pmatrix} E_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 \\ 0 & 0 & E_3 & 0 \\ 0 & 0 & 0 & E_4 \end{pmatrix} \cdot \begin{pmatrix} \frac{e^{-i\omega_1 t}}{\sqrt{3}} \\ \frac{e^{-i\omega_2 t}}{2} \\ \frac{e^{-i\omega_3 t}}{\sqrt{6}} \\ \frac{e^{-i\omega_4 t}}{2} \end{pmatrix}$$

Don't forget to conjugate!

$$= \left(\frac{e^{+i\omega_1 t}}{\sqrt{3}}, \frac{e^{+i\omega_2 t}}{2}, \frac{e^{+i\omega_3 t}}{\sqrt{6}}, \frac{e^{+i\omega_4 t}}{2} \right) \begin{pmatrix} E_1 \cdot \frac{e^{-i\omega_1 t}}{\sqrt{3}} \\ E_2 \frac{e^{-i\omega_2 t}}{2} \\ E_3 \frac{e^{-i\omega_3 t}}{\sqrt{6}} \\ E_4 \frac{e^{-i\omega_4 t}}{2} \end{pmatrix}$$

$\boxed{e^{i\omega_1 t} \cdot e^{-i\omega_1 t}}$ etc

$$\boxed{\langle \mathbb{H} \rangle_t = \frac{1}{3} \cdot E_1 \cdot 1 + \frac{1}{4} E_2 + \frac{1}{6} E_3 + \frac{1}{4} E_4}$$

(3.16) Consider a system whose Hamiltonian $\hat{H}$ and an operator $\hat{A}$ are given by

$$\mathbb{H} = \varepsilon_0 \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 2i \\ 0 & -2i & 0 \end{pmatrix} \qquad \mathbb{A} = a_0 \begin{pmatrix} 0 & -i & 0 \\ i & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

a) If we measure energy, what values will we obtain?

- Eigenevalues of $\mathbb{H}$
- Calculate (you may use Maple)

$\det[\mathbb{H} - \lambda \mathbb{I}] = 0$

$$\begin{vmatrix} -\lambda & -i & 0 \\ i & -\lambda & 2i \\ 0 & -2i & -\lambda \end{vmatrix} = 0 \iff -\lambda \begin{vmatrix} -\lambda & 2i \\ -2i & -\lambda \end{vmatrix} - (-i) \begin{vmatrix} i & 2i \\ 0 & -\lambda \end{vmatrix} = 0$$

Expand by minors

$$-\lambda(\lambda^2 - 4) + i(-i\lambda) = 0$$
$$\Updownarrow$$
$$-\lambda(\lambda^2-4) + \lambda = 0$$
$$\Updownarrow$$
$$(-\lambda)[(\lambda^2-4) - 1] = 0$$
$$\lambda = 0, \quad \lambda = \pm\sqrt{5}$$

Energy eigenvalues are $0, \pm\sqrt{5}\,\varepsilon_0$

So, after a measurement of $\mathbb{H}$ we may obtain 0, $\sqrt{5}\,\varepsilon_0$, or $-\sqrt{5}\,\varepsilon_0$

(3.16) b) Suppose we measure $\mathbb{H}$ and obtain $\sqrt{5}\,\varepsilon_0$. Immediately after, we measure $\mathbb{A}$. What values will we obtain for $\mathbb{A}$, and with what probabilities?

First, find all the eigenvalues and eigenvectors (normalized) of $\mathbb{H}$ and $\mathbb{A}$. You may use Maple. Note that Maple does not give normalized eigenvectors, so you have to do that.

From Maple

		1	2	3
$\mathbb{H}$:	Eigenvalues	0	$+\sqrt{5}\,\varepsilon_0$	$-\sqrt{5}\,\varepsilon_0$
		$\updownarrow$	$\updownarrow$	$\updownarrow$
	Eigenvectors $\lvert\phi_i\rangle$	$\frac{1}{\sqrt{5}}\begin{pmatrix}-2\\0\\-1\end{pmatrix}$	$\begin{pmatrix}1/\sqrt{10}\\i/\sqrt{2}\\\sqrt{2/5}\end{pmatrix}$	$\begin{pmatrix}1/\sqrt{10}\\-i/\sqrt{2}\\\sqrt{2/5}\end{pmatrix}$
$\mathbb{A}$	Eigenvalues	$2a_0$	0	$-a_0$
	Eigenvectors $\lvert\chi_i\rangle$	$\frac{1}{\sqrt{6}}\begin{pmatrix}-i\\2\\1\end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix}i\\0\\1\end{pmatrix}$	$\frac{1}{\sqrt{3}}\begin{pmatrix}-i\\-i\\1\end{pmatrix}$

After $\mathbb{H}$ we obtain $+\sqrt{5}\,\varepsilon_0$, so we are in the state

$$\lvert\psi\rangle = \lvert\phi_2\rangle = \begin{pmatrix}1/\sqrt{10}\\i/\sqrt{2}\\\sqrt{2/5}\end{pmatrix}$$

Eigenstate of $\mathbb{H}$

What eigenstates of $\mathbb{A}$ is this state made of?

$$\lvert\phi_2\rangle = \mathbb{1}\lvert\phi_2\rangle = \sum_i \lvert\chi_i\rangle\langle\chi_i\vert\phi_2\rangle$$

Eigenstates of $\mathbb{A}$

This is a change of basis

Calculate $\langle x_1 | \phi_2 \rangle$

$\langle x_2 | \phi_2 \rangle$

$\langle x_3 | \phi_2 \rangle$

$$\langle x_1 | \phi_2 \rangle = \frac{1}{\sqrt{6}}(i, 2, 1)\begin{pmatrix} 1/\sqrt{10} \\ i/\sqrt{2} \\ \sqrt{2/5} \end{pmatrix} = \frac{i}{\sqrt{60}} + \frac{2i}{\sqrt{12}} + \sqrt{\frac{2}{30}}$$

$$= \boxed{\frac{1}{\sqrt{15}} + i\left(\frac{1}{\sqrt{60}} + \frac{1}{\sqrt{3}}\right)}$$

$$\langle x_2 | \phi_2 \rangle = \frac{1}{\sqrt{2}}(-i, 0, 1)\begin{pmatrix} 1/\sqrt{10} \\ i/\sqrt{2} \\ \sqrt{2/5} \end{pmatrix}$$

$$= \frac{-i}{\sqrt{20}} + 0 + \frac{1}{\sqrt{5}} = \boxed{\frac{1}{\sqrt{5}} - \frac{i}{\sqrt{20}}}$$

$$\langle x_3 | \phi_2 \rangle = \frac{1}{\sqrt{3}}(+i, -1, 1)\begin{pmatrix} 1/\sqrt{10} \\ i/\sqrt{2} \\ \sqrt{2/5} \end{pmatrix} = \frac{i}{\sqrt{30}} - \frac{i}{\sqrt{6}} + \sqrt{\frac{2}{15}}$$

$$= \boxed{\sqrt{\frac{2}{15}} + \left(\frac{1}{\sqrt{30}} - \frac{1}{\sqrt{6}}\right)i}$$

	Probability		Probability of measuring	
$P_1 = \lvert\langle x_1 \lvert \phi_2 \rangle\rvert^2 =$	$\boxed{.5657}$	$\leftrightarrow$	$+2a_0$	is 56.57%
$P_2 = \lvert\langle x_2 \lvert \phi_2 \rangle\rvert^2 =$	$\boxed{.25}$	$\leftrightarrow$	$0a_0$	25%
$P_3 = \lvert\langle x_3 \lvert \phi_2 \rangle\rvert^2 =$	$\boxed{.1843}$	$\leftrightarrow$	$-a_0$	18.43%

c) Calculate the expectation value of A for $|\phi_2\rangle$.

$$\langle A \rangle = a_1 \cdot P_1 + a_2 P_2 + a_3 P_3$$

$$= 2a_0 \cdot 56.57\% + 0 \cdot 25\% + -a_0 \cdot 18.43\%$$

$$= \boxed{0.9471 \cdot a_0}$$

(3.17) Given $\mathbb{H} = \varepsilon_0 \left(\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \\ \hline 0 & 0 & -1 \end{array} \right)$

$|\psi(0)\rangle = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$ (normalized ✓)

where ε_0 = energy

a) What values will we obtain when $\mathbb{H}$ is measured, and with what probabilities?

From Maple:

$0 \cdot \varepsilon_0 \leftrightarrow \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad -\varepsilon_0 \leftrightarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad 2\varepsilon_0 \leftrightarrow \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Eigenvalues of $\mathbb{H}$

Normalize

$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\quad \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

Normalized eigenvectors ($|\phi_i\rangle$) $\quad i=1 \quad i=2 \quad i=3$

So measurements of $\mathbb{H}$ can give $0, -\varepsilon_0, +2\varepsilon_0$

if initial state is $|\psi(0)\rangle$ is

The probability of each state ~~is~~

$$\boxed{P_i = |\langle \phi_i | \psi(0) \rangle|^2}$$

So $|\langle \phi_1 | \psi(0)\rangle|^2 = \left| \frac{1}{\sqrt{2}} (1,1,0) \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{12}} (1+1+0) \right|^2 = \boxed{\frac{1}{3}}$

$|\langle \phi_1 | \psi(0)\rangle|^2 = \left| (0,0,1) \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right|^2 = \left| \frac{2}{\sqrt{6}} \right|^2 = \boxed{\frac{2}{3}}$

$|\langle \phi_2 | \psi(0)\rangle|^2 = \left| \frac{1}{\sqrt{2}} (-1,1,0) \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right|^2 = \left| \frac{1}{\sqrt{12}} (0) \right|^2 = \boxed{0}$

$|\langle \phi_3 | \psi(0)\rangle|^2 =$

So $0 \leftrightarrow 33\%$

$-\varepsilon_0 \leftrightarrow 66\%$

$2\varepsilon_0 \leftrightarrow 0\% \leftrightarrow$ Starting in $|\psi(0)\rangle$, one cannot get state $|\phi_3\rangle$!

3.17 b) Calculate the expectation value of $\mathbb{H}$

$$\langle \mathbb{H} \rangle = \varepsilon_1 \cdot P_1 + \varepsilon_2 \cdot P_2 + \varepsilon_3 \cdot P_3$$

$$= 0 \cdot \frac{1}{3} + (-\varepsilon_0) \cdot \frac{2}{3} + (2\varepsilon_0) \cdot 0$$

$$= \boxed{-.667 \cdot \varepsilon_0}$$

3.18 Given:

$$|\psi(t)\rangle = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$$

$$\mathbb{A} = \frac{a}{\sqrt{2}} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad \mathbb{B} = b \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

a) We measure $\mathbb{A}$ first, then $\mathbb{B}$ immediately after.

Find the probability of measuring $\sqrt{2}a$ for $\mathbb{A}$ and $-b$ for $\mathbb{B}$

- First use Maple to write the eigenvectors and eigenvalues of $\mathbb{A}$, $\mathbb{B}$

		1	2	3
$\mathbb{A}$	Eigenvalues	$\sqrt{2}a$	$\sqrt{2}a$	0
$\lvert \phi\rangle$	Eigenvectors	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$
$\mathbb{B}$	Eigenvalues	b	b	$-b$
$\lvert \chi\rangle$	Eigenvectors	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

- Normalize $|\psi(t)\rangle = \frac{1}{\sqrt{35}}\begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$

After measuring A, probabilities of being in eigenstates $|\phi_1\rangle, |\phi_2\rangle, |\phi_3\rangle$ of A are:

value $\sqrt{2}a$

$$\left\{\begin{array}{l} P_1 = |\overbrace{\langle\phi_1|\psi(t)\rangle}^{c_i}|^2 = \left|\frac{1}{\sqrt{2}}(0,1,1)\frac{1}{\sqrt{35}}\begin{pmatrix}5\\1\\3\end{pmatrix}\right|^2 = \boxed{\frac{16}{70}} = 22.86\% \\ P_2 = |\langle\phi_2|\psi(t)\rangle|^2 = \left|(1,0,0)\frac{1}{\sqrt{35}}\begin{pmatrix}5\\1\\3\end{pmatrix}\right|^2 = \boxed{\frac{25}{35}} = 71.43\% \end{array}\right.$$

0

$$\left\{ P_3 = |\langle\phi_3|\psi(t)\rangle|^2 = \left|\frac{1}{\sqrt{2}}(0,1,-1)\frac{1}{\sqrt{35}}\begin{pmatrix}5\\1\\3\end{pmatrix}\right|^2 = \boxed{\frac{4}{70}} = 5.71\% \right.$$

The state you are in after measuring A is

$$|\psi(t)\rangle_{after} = \frac{c_1|\phi_1\rangle + c_2|\phi_2\rangle}{\sqrt{c_1^2+c_2^2}}$$

eigenvectors of $\sqrt{2}a$

renormalization factor

After measuring B on this state, what is the probability of getting measurement $-b$?

$-b$ is the eigenvalue of state $|\chi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\-1\\1\end{pmatrix}$

The probability of this state after measurement of $\sqrt{2}a$ is

$$|\langle\chi_3|\psi(t)\rangle_{after}|^2 = \left|\frac{c_1\overbrace{\langle\chi_3|\phi_1\rangle}^{0} + c_2\overbrace{\langle\chi_3|\phi_2\rangle}^{0}}{\sqrt{c_1^2+c_2^2}}\right|^2$$

$$= |0+0|^2$$

$$= \boxed{0}$$

$$\boxed{\text{Thus it is not possible to obtain } -b \text{ from } B \text{ after } \sqrt{2}a \text{ from } A.}$$

3.18 (b) Now find probability of

$$|\psi(t)\rangle \longrightarrow \boxed{B} \xrightarrow{-b} \boxed{A} \xrightarrow{+\sqrt{2}a}$$

Measurement of $-b$ means the system is in state $|x_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$

Since this state is orthogonal to $|\phi_1\rangle$ and $|\phi_2\rangle$ measurement of A can only yield $0 \cdot a_0$

Same as part (a).

(c) Because $[A,B]=0$ A and B are compatible. Thus they share eigenvectors, and the order of measurement does not matter.

3.21 Given: $|\psi(0)\rangle = \begin{pmatrix} 4-i \\ -2+5i \\ 3+2i \end{pmatrix}$

$$\mathbb{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 3 & 3 \\ 0 & 3 & 0 \end{pmatrix}\varepsilon_0$$

a) If $\mathbb{H}$ is measured, what values will you obtain and with what probabilities?

From Maple		1	2	3
	Eigenvalues	$-\sqrt{2}\,\varepsilon_0$	0	$\frac{5}{\sqrt{2}}\varepsilon_0$
	Eigenvectors $\lvert\phi_i\rangle$	$\frac{3}{\sqrt{14}}\begin{pmatrix} -i/3 \\ -2/3 \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{10}}\begin{pmatrix} 3i \\ 0 \\ 1 \end{pmatrix}$	$\frac{3}{\sqrt{35}}\begin{pmatrix} -i/3 \\ 5/3 \\ 1 \end{pmatrix}$

Measurement of H may yield $-\sqrt{2}\,\varepsilon_0$, 0, or $\frac{5}{\sqrt{2}}\varepsilon_0$

Probabilities of each:

$$P_1 = \frac{|\langle \phi_1 | \psi(0)\rangle|^2}{\langle \psi(0)|\psi(0)\rangle} = \frac{\left|\frac{3}{\sqrt{14}}\left(\frac{i}{3}, -\frac{2}{3}, 1\right)\begin{pmatrix} 4-i \\ -2+5i \\ 3+2i \end{pmatrix}\right|^2}{4^2+1^2+2^2+5^2+3^2+2^2}$$

$$= \frac{\left|\frac{3}{\sqrt{14}}\left(\frac{4}{3}i + \frac{1}{3} + \frac{4}{3} - \frac{10}{3}i + 3 + 2i\right)\right|^2}{16+1+4+25+9+4}$$

$$= \frac{\frac{9}{14}\cdot\left|\frac{1}{3}+\frac{4}{3}+3 + i\left(\frac{4}{3}-\frac{10}{3}+2\right)\right|^2}{59}$$

$$= \frac{9}{14\cdot 59}\cdot\left|\frac{14}{3} + i\left(-\frac{2}{3}\right)\right|^2$$

$$= \boxed{0.2421} = 24.2\%$$

$$P_2 = \frac{|\langle \phi_2 | \psi(0)\rangle|^2}{\langle \psi(0)|\psi(0)\rangle} = \frac{\left|\frac{1}{\sqrt{10}}(-3i, 0, 1)\begin{pmatrix} 4-i \\ -2+5i \\ 3+2i \end{pmatrix}\right|^2}{59}$$

$$= \frac{\frac{1}{10}\left|\left(-12i - \cancel{3} + 0 + \cancel{3} + 2i\right)\right|^2}{59}$$

$$= \frac{|-\cancel{3}+\cancel{3} - 10i|^2}{590} = \frac{100}{590} = \boxed{0.1695} = 17.0\%$$

$$P_3 = \frac{|\langle \phi_3 | \psi(0)\rangle|^2}{\langle \psi(0)|\psi(0)\rangle} = \frac{\left|\frac{3}{\sqrt{35}}\left(i/3, 5/3, 1\right)\begin{pmatrix} 4-i \\ -2+5i \\ 3+2i \end{pmatrix}\right|^2}{59}$$

$$= \frac{\frac{9}{35}}{59}\left|\left(\frac{4}{3}i + \frac{1}{3} + \frac{-10}{3} + \frac{25}{3}i + 3 + 2i\right)\right|^2 = \frac{\frac{9}{35}\left|0 + \frac{35}{3}i\right|^2}{59}$$

$$= \boxed{0.5932} = 59.3\%$$

b) Find the state of the system at a later time t

$$|\psi(0)\rangle = \mathbb{1}\cdot|\psi(0)\rangle = \Big(\sum_i \underbrace{|\phi_i\rangle\langle\phi_i|\Big)|\psi(0)\rangle}_{\text{associate}}$$

$$= \sum_i \underbrace{\langle\phi_i|\psi(0)\rangle}_{\substack{\text{calculated}\\ \text{in (a)}}}\cdot|\phi_i\rangle$$

energy eigenstate

$$= \sum_i c_i\,|\phi_i\rangle$$

Then $$|\psi(t)\rangle = \sum_i c_i\,|\phi_i\rangle\, e^{-i\omega_i t}$$

$$= \frac{3}{\sqrt{14\cdot 59}}\left(\frac{14}{3}+(-)i\frac{2}{3}\right)|\phi_1\rangle e^{-i\omega_1 t}$$

$$+ \frac{1}{\sqrt{59}}(-i)\,|\phi_2\rangle e^{-i\omega_2 t}$$

$$+ \frac{3}{\sqrt{35\cdot 59}}(35\,i)\,|\phi_3\rangle e^{-i\omega_3 t}$$

where $\omega_1 = -\dfrac{\sqrt{2}\,\varepsilon_0}{\hbar}$

$\omega_2 = 0$

$\omega_3 = \dfrac{5\varepsilon_0/\sqrt{2}}{\hbar}$

c) Find the total energy of the system at time $t=0$

My guess is he means to find $\langle H\rangle_{t=0}$ over the state $|\psi(0)\rangle$, then $|\psi(t)\rangle$

$$\langle H\rangle_{t=0} = -\sqrt{2}\cdot\varepsilon_0\cdot \overset{.2421}{P_1} + 0\cdot \overset{.1695}{P_2} + \frac{5}{\sqrt{2}}\varepsilon_0\cdot \overset{.5932}{P_3}$$

$$= \boxed{1.75\,\varepsilon_0}$$

It's the same for $\langle\psi(t)|H|\psi(t)\rangle$

d) Does $\mathbb{H}$ form a complete set of commuting operators?

I am a little puzzled by this question, since only one operator was given. $\mathbb{H}$ commutes w/ itself, of course, but we need a second operator to discuss this question meaningfully.

3.23 Given a Hamiltonian

$$\mathbb{H} = \varepsilon \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

(a) Find the eigenvalues

	Eigenvalues	Eigenvectors ($\|\phi_i\rangle$)
1	$+\varepsilon$	$\leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} -i \\ 1 \end{pmatrix}$
2	$-\varepsilon$	$\leftrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} i \\ 1 \end{pmatrix}$

(b) If the system is initially in $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = |\psi\rangle$ find probabilities

$$P_1 = |\langle \phi_1 | \psi(0)\rangle|^2 = \left|\frac{1}{\sqrt{2}}(i, 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|^2$$

$$= \frac{1}{2}|i|^2$$

$$= \boxed{\frac{1}{2}}$$

$$P_2 = |\langle \phi_2 | \psi(0)\rangle|^2 = \left|\frac{1}{\sqrt{2}}(-i, 1)\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|^2$$

$$= \boxed{\frac{1}{2}}$$

So 50% for $+\varepsilon$
50% for $-\varepsilon$

(c) Find

$\langle \mathbb{H} \rangle$ and $\Delta \mathbb{H} = \sqrt{\langle \mathbb{H}^2 \rangle - \langle \mathbb{H} \rangle^2}$

$$\langle \mathbb{H} \rangle = +\varepsilon \cdot \frac{1}{2} + (-\varepsilon) \cdot \frac{1}{2} = \boxed{0}$$

$$\langle \mathbb{H}^2 \rangle = \varepsilon^2 \cdot \frac{1}{2} + \varepsilon^2 \cdot \frac{1}{2} = \varepsilon^2$$

$$\text{So } \Delta \mathbb{H} = \sqrt{\varepsilon^2 - 0} = \boxed{\varepsilon}$$

(d) Find $|\psi(t)\rangle$

$$|\psi(0)\rangle = \frac{i}{\sqrt{2}} |\phi_1\rangle + \frac{-i}{\sqrt{2}} |\phi_2\rangle$$

eigenstates of $\mathbb{H}$

$$\boxed{\text{Then } |\psi(t)\rangle = \frac{i}{\sqrt{2}} |\phi_1\rangle e^{-i\omega_1 t} + \frac{-i}{\sqrt{2}} |\phi_2\rangle e^{-i\omega_2 t}}$$

$$\omega_1 = \frac{\varepsilon}{\hbar}$$

$$\omega_2 = -\frac{\varepsilon}{\hbar}$$

(3.27) Consider a system whose initial state is

$$|\psi(0)\rangle = \frac{A}{\sqrt{12}}|\phi_1\rangle + \frac{1}{\sqrt{6}}|\phi_2\rangle + \frac{2}{\sqrt{12}}|\phi_3\rangle + \frac{1}{2}|\phi_4\rangle$$

where $\{|\phi_i\rangle\}$ is complete and orthonormal

$A \equiv$ real constant

a) Find A so $|\psi(0)\rangle$ is normalized.

$$\langle\psi(0)|\psi(0)\rangle = 1 = \frac{A^2}{12} + \frac{1}{6} + \frac{4}{12} + \frac{1}{4}$$

$$= \frac{A^2}{12} + \frac{2}{12} + \frac{4}{12} + \frac{3}{12} = \frac{A^2}{12} + \frac{9}{12} = 1$$

so $\boxed{A = \sqrt{3}}$

b) Find $|\psi(t)\rangle$ at any later time if the energies corresponding to $|\phi_i\rangle$ are

$$\begin{array}{cccc} |\phi_1\rangle & |\phi_2\rangle & \cdots & |\phi_4\rangle \\ \updownarrow & \updownarrow & & \updownarrow \\ E_1 & E_2 & & E_4 \end{array}$$

OK, the author did not mention at the beginning that the $|\phi_i\rangle$ were eigenstates of the Hamiltonian, but he has now.

Then

$$\boxed{\begin{aligned} |\psi(t)\rangle = {} & \sqrt{\frac{3}{12}}\, e^{-i\omega_1 t}|\phi_1\rangle + \frac{1}{\sqrt{6}} e^{-i\omega_2 t}|\phi_2\rangle \\ & + \frac{2}{\sqrt{12}} e^{-i\omega_3 t}|\phi_3\rangle + \frac{1}{2} e^{-i\omega_4 t}|\phi_4\rangle \end{aligned}}$$

$\omega_1 = \frac{E_1}{\hbar}$ ... etc

c) Determine the probability of finding the system in state $|\varphi_2\rangle$

~~Check~~ $\cancel{\langle\Psi(t)|\Psi(t)\rangle^2 = 1}$

[illegible]

$$P_2 = |\langle\phi_2|\psi(t)\rangle|^2$$

$$= \left|\frac{1}{\sqrt{6}}e^{-i\omega_2 t}\right|^2$$

$$= \boxed{\frac{1}{6}} = 16.7\%$$