Quantum Mechanics

Problems 3.1, 3.2, 3.3

(3.1) A particle in an infinite potential well wi sides at $x=0$ and $x=a$ has the following wavefunction

$$\Psi(x) = \frac{1}{\sqrt{5a}} \sin\left(\frac{\pi x}{a}\right) + \frac{2}{\sqrt{5a}} \sin\left(\frac{3\pi x}{a}\right)$$

a) Find the possible results of measuring the system's energy and the corresponding probabilities.

- Write on the basis of the Hamiltonian $|\Psi_n\rangle$

$$|\Psi\rangle = \frac{1}{\sqrt{10}} \underbrace{\sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right)}_{\Downarrow} + \frac{2}{\sqrt{10}} \underbrace{\sqrt{\frac{2}{a}} \sin\left(\frac{3\pi x}{a}\right)}_{\Downarrow}$$

$$= \frac{1}{\sqrt{10}} |\Psi_1\rangle + \frac{2}{\sqrt{10}} |\Psi_3\rangle$$

- Normalize

$$\langle\Psi|\Psi\rangle = c^2\left(\frac{1}{10} + \frac{4}{10}\right) = \frac{c^2}{2} \Rightarrow c = \sqrt{2}$$

- Thus the normalize $|\Psi\rangle$ is

$$|\Psi\rangle = \frac{1}{\sqrt{5}} |\Psi_1\rangle + \frac{2}{\sqrt{5}} |\Psi_3\rangle$$

- The possible energy measurements are E_1 and E_3 where $E_n = \frac{n^2\pi^2}{a^2} \cdot \frac{\hbar^2}{2m}$ $\quad n = 1, 3$

- The probabilities are

$$P_1 = |\langle\Psi_1|\Psi\rangle|^2 = \frac{1}{5} = 20\%$$

$$P_3 = |\langle\Psi_3|\Psi\rangle|^2 = \frac{4}{5} = 80\%$$

b) Find the form of the wavefunction after such a measurement.

If you find E_1, then you are in state $|\psi_1\rangle$

" E_3, $|\psi_3\rangle$ "

c) If the energy is measured again immediately afterwards, what are the relative probabilities of the possible outcomes?

Once in the eigenstate, the system would remain in that state for all other measurements of H.

(3.2) Let $\psi_n(x)$ denote the orthonormal states of a system corresponding to the energy E_n. Suppose that the normalized w.f. at time zero is $\psi(x,0)$ and suppose that a measure of energy yields E_1 with probability $\frac{1}{2}$, E_2 w/ probability $3/8$, and E_3 w/ probability $1/8$.

a) Write the most general expansion for $\psi(x,0)$ consistent w/ this information

$$|\psi_n\rangle = \frac{1}{\sqrt{2}} \cdot e^{i\phi_1} |\psi_1\rangle + \sqrt{\frac{3}{8}} \cdot e^{i\phi_2} |\psi_2\rangle$$
$$+ \frac{1}{\sqrt{8}} \cdot e^{i\phi_3} |\psi_3\rangle$$

where ϕ_1, ϕ_2, ϕ_3 are real phase shifts. Note that in each case $c \cdot c^* = |c|^2$ gives the correct probability.

b) What is the expansion for $\Psi(x,t)$?

Since the wave function is written in the basis of $\mathbb{H}$, the rule is to multiply each term by $e^{-i\omega t}$

$$|\psi(x,t)\rangle = \frac{1}{\sqrt{2}} \cdot e^{i\phi_1} \cdot e^{-i\omega_1 t} |\psi_1\rangle$$
$$+ \sqrt{\frac{3}{8}} \cdot e^{i\phi_2} \cdot e^{-i\omega_2 t} |\psi_2\rangle$$
$$+ \frac{1}{\sqrt{8}} \cdot e^{i\phi_3} \cdot e^{-i\omega_3 t} |\psi_3\rangle$$

c) Show that the expectation value of $\mathbb{H}$ does not change w/ time.

$$\langle \mathbb{H} \rangle = \langle \psi(t) | \mathbb{H} | \psi(t) \rangle$$
$$= \left(\frac{1}{\sqrt{2}}\right)^2 \cdot E_1 + \left(\sqrt{\frac{3}{8}}\right)^2 \cdot E_2 + \left(\frac{1}{\sqrt{8}}\right)^2 \cdot E_3 \quad \text{for all } t$$

So $\langle \mathbb{H} \rangle$ does not change w/ time.

(3.3) Consider a neutron in an infinite square well of width $a = 8\,\text{fm}$. At time $t=0$ the neutron is in the state

$$\psi(x,0) = \sqrt{\frac{4}{7a}} \sin\left(\frac{\pi x}{a}\right) + \sqrt{\frac{2}{7a}} \sin\left(\frac{2\pi x}{a}\right) + \sqrt{\frac{8}{7a}} \sin\left(\frac{3\pi x}{a}\right)$$

a) If energy is measured, what values will be observed and w/ what probabilities?

Rewrite

$$|\psi(0)\rangle = \sqrt{\frac{2}{7}} |\psi_1\rangle + \frac{1}{\sqrt{7}} |\psi_2\rangle + \frac{2}{\sqrt{7}} |\psi_3\rangle$$

Normalize

$$\langle \psi(0) | \psi(0) \rangle = 1 = c^2\left(\frac{2}{7} + \frac{1}{7} + \frac{4}{7}\right) = 1, \text{ so } c = 1$$

The normalized $|\psi(0)\rangle$ on the basis of $\mathbb{H}$ is therefore

$$|\psi(0)\rangle = \sqrt{\frac{2}{7}}\,|\psi_1\rangle + \frac{1}{\sqrt{7}}\,|\psi_2\rangle + \frac{2}{\sqrt{7}}\,|\psi_3\rangle$$

The possible energy measurements are the eigenvalues

$$E_1,\ E_2,\ E_3 \quad \text{where } E_n = \frac{\hbar^2}{2m}\cdot\frac{n^2\pi^2}{a^2}$$

$$\begin{array}{lccc} & \updownarrow & \updownarrow & \updownarrow \\ \text{Probability:} & \frac{2}{7} & \frac{1}{7} & \frac{4}{7} \end{array}$$

b) What is the average energy obtained if the experiment is repeated many times?

$$\langle H \rangle = \frac{2}{7}\cdot E_1 + \frac{1}{7}\cdot E_2 + \frac{4}{7}\cdot E_3$$

c) Using the uncertainty principle, estimate the order of magnitude of the neutron's speed in this well as a function of the speed of light.

The position-momentum uncertainty principle is

$$\Delta x\cdot\Delta p \gtrsim \frac{\hbar}{2}$$

$$\Delta x\cdot\Delta p \sim \hbar$$

$$\Delta x\cdot m\cdot\Delta v \sim \hbar$$

$\downarrow$ Take the uncertainty in x to be the size of the well.
a

Then

$$\Delta v \sim \frac{\hbar}{ma}$$

Take the neutron's speed to be on the order of Δv

$$v \sim \Delta v = \frac{\hbar}{am}\cdot\frac{c^2}{c^2} = \boxed{\frac{\hbar c\cdot c}{a\, mc^2}} = \frac{197\,\text{MeV}\cdot\text{fm}\cdot c}{8\,\text{fm}\ 939\,\text{MeV}} \sim \boxed{3\%\,c}$$