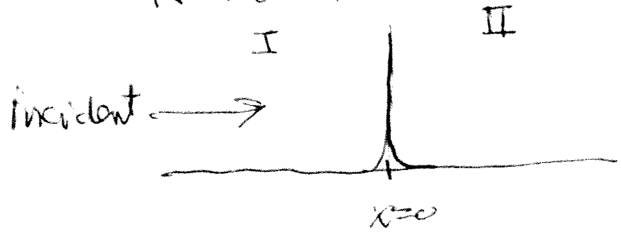
(4.23) A particle of mass m is subjected to a repulsive delta function potential $V(x) = V_0\,\delta(x)$ where $V_0 > 0$ (V_0 has units of Energy × Distance). Find the reflection and transmission coefficients R and T.



The Schrödinger eqn is:

$$-\frac{\hbar^2}{2m}\psi''(x) + V(x)\psi(x) = E\psi(x)$$

This is
$$\psi''(x) = -k^2\psi(x)$$
everywhere except at the origin

$$\psi_I = A e^{ikx} + B e^{-ikx}$$
$$\psi_{II} = C e^{ikx}$$

Since $\psi_I(0) = \psi_{II}(0)$

$$A + B = C \quad \ldots\ldots (1)$$

What is the condition on the derivative?

$$\psi''(x) = -\frac{2m(E - V(x))}{\hbar^2}\psi(x).$$

Take the integral of both sides symmetrically around a small range $\pm\varepsilon$

$$\int_{-\varepsilon}^{+\varepsilon} \psi''(x)\,dx = -\frac{2mE}{\hbar^2}\int_{-\epsilon}^{+\epsilon} \psi(x)\,dx + \frac{2mV_0}{\hbar^2}\int_{-\epsilon}^{+\epsilon} \psi(x)\delta(x)\,dx$$

$$\psi'(+\varepsilon) - \psi'(-\varepsilon) = 0 + \beta\,\psi(0) \qquad \beta \equiv \frac{2mV_0}{\hbar^2}$$

$$\boxed{\psi'_{II}(0) = \psi'_{I}(0) + \beta\,\psi_I(0)} \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

This gives $\quad C\,ik = A\,ik - B\,ik + \beta(A+B)$

$$= A(ik+\beta) + B(\beta - ik)$$

or $\quad \frac{C}{A}\,ik = (ik+\beta) + \frac{B}{A}(\beta - ik) \quad \ldots (2)$

Solving (1) and (2) to find $\left|\frac{C}{A}\right|^2$ and $\left|\frac{B}{A}\right|^2$, we will then calculate

$$R = \frac{\frac{\hbar k_I}{m}|B|^2}{\frac{\hbar k_I}{m}|A|^2} = \left|\frac{B}{A}\right|^2$$

$$T = \frac{\frac{\hbar k_{II}}{m}|C|^2}{\frac{\hbar k_I}{m}|A|^2} = \left|\frac{C}{A}\right|^2 \quad \text{since } k_I = k_{II}$$

From (1)

$$\frac{C}{A} = 1 + \frac{B}{A}$$

From (2)

$$\frac{C}{A} = \left(1 - i\frac{\beta}{k}\right) + \frac{B}{A}\left(-i\frac{\beta}{k} - 1\right) = \left(1 - i\frac{\beta}{k}\right) - \frac{B}{A}\left(1 + i\frac{\beta}{k}\right)$$

Subtracting:

$$0 = 1 + \frac{B}{A} - \left(1 - i\frac{\beta}{k}\right) + \frac{B}{A}\left(1 + i\frac{\beta}{k}\right)$$

$$0 = i\frac{\beta}{k} + \frac{B}{A}\left(2 + i\frac{\beta}{k}\right) \Rightarrow \boxed{\frac{B}{A} = \frac{-i\frac{\beta}{k}}{2 + i\frac{\beta}{k}}}$$

Thus $R = \left|\frac{B}{A}\right|^2 = \frac{\beta^2/k^2}{4+\beta^2/k^2} = \frac{1}{1+4\frac{k^2}{\beta^2}}$

$$\beta^2 = \left(\frac{2mV_0}{\hbar^2}\right)^2$$

$$k^2 = \frac{2mE}{\hbar^2}$$

$$= \frac{1}{1+4\frac{2mE/\hbar^2}{4m^2V_0^2/\hbar^4}}$$

$$\boxed{R = \frac{1}{1+\frac{2\hbar^2E}{mV_0^2}}}$$

Thus

$$\boxed{T = \left|\frac{C}{A}\right|^2 = 1 - \left|\frac{B}{A}\right|^2 = \frac{\frac{2\hbar^2E}{mV_0^2}}{1+\frac{2\hbar^2E}{mV_0^2}}}$$

(4.31) A particle is initially in the ground state in an infinite square well with sides $x=0, a$. If the wall of the box at $x=a$ is suddenly moved to $3a$, calculate the probability of finding the particle in

a) the ground state of the new box

For a 1D infinite square well, $\psi_n = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$

So $\psi_{initial} = \sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)$

$\psi_{final} = \sqrt{\frac{2}{3a}}\sin\left(\frac{\pi x}{3a}\right)$

Probability of being in final state ψ after initial state ϕ

$$P = |\langle\psi|\phi\rangle|^2 = \left|\int \psi^*(x)\,\phi(x)\,dx\right|^2$$

$$= \left|\int_0^a \sqrt{\frac{2}{3a}}\cdot\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{3a}\right)dx\right|^2$$

$$= \frac{4}{a^2}\left|\frac{1}{\sqrt{3}}\int_0^a \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi x}{3a}\right)dx\right|^2$$

$$= \frac{4}{3a^2}\left|\frac{3a}{\pi}\int_0^{\pi/3} \sin\alpha\cdot\sin 3\alpha\; d\alpha\right|^2$$

$$\alpha \equiv \frac{\pi x}{3a}$$

$$= \frac{4}{3\cancel{a^2}}\cdot\frac{9\cancel{a^2}}{\pi^2}\cdot\left|\int_0^{\pi/3}\sin\alpha\,\sin 3\alpha\,d\alpha\right|^2$$

$$= \frac{12}{\pi^2}\left|\int_0^{\pi/3}\sin\alpha\,\sin 3\alpha\,d\alpha\right|^2 = \frac{12}{\pi^2}\left|\frac{3\sqrt{3}}{16}\right|^2$$

$$= \frac{12\cdot 9\cdot 3}{\pi^2\cdot 2^8}$$

$$= 0.128 = \boxed{12.8\%}$$

b) The first excited state of the new box.

$$P = |\langle\psi|\phi\rangle|^2 = \left| \int_0^a \underbrace{\sqrt{\frac{2}{3a}}\, \sin\left(\frac{2\pi x}{3a}\right)}_{\text{final}} \underbrace{\sqrt{\frac{2}{a}}\sin\left(\frac{\pi x}{a}\right)}_{\text{initial}} dx \right|^2$$

$$= 0.328 = \boxed{32.8\%}$$ (done in Maple)

c) Now assume the particle was initially in the first excited state of the ~~new~~ old box, and you want the probability of being in the first excited state of the new box.

$$\psi_{initial} = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right) = |\phi\rangle$$

$$\psi_{final} = \sqrt{\frac{2}{3a}} \sin\left(\frac{2\pi x}{3a}\right) = |\psi\rangle$$

$$P = |\langle\psi|\phi\rangle|^2 = \left| \int_0^a \sqrt{\frac{2}{3a}} \sin\left(\frac{2\pi x}{3a}\right) \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right) dx \right|^2$$

$$= 0.0321 = \boxed{3.21\%}$$ (from Maple)

Problem 4.27 c) Solutions of the eqn -beta=gamma*tan(beta)

```
> with(plots):
> gam:=2;
```

$$gam := 2 \tag{1.1}$$

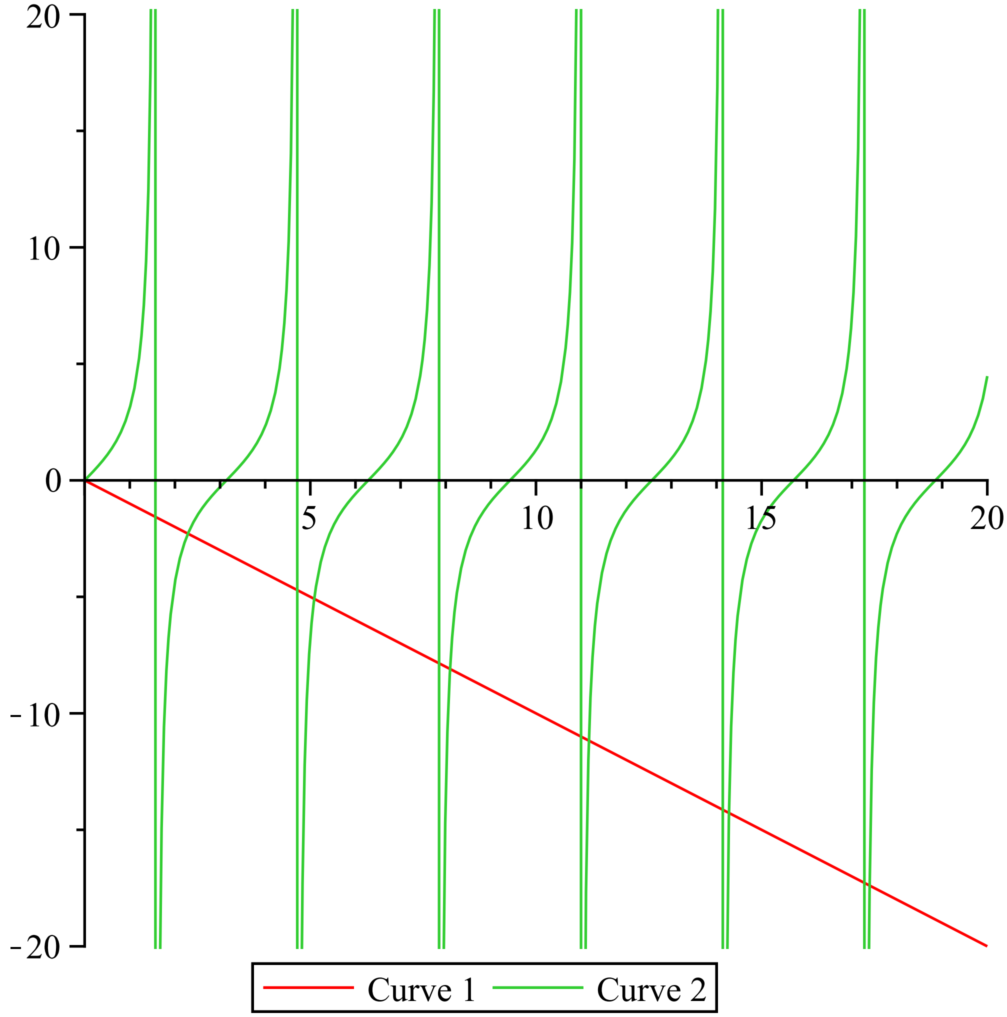
```
> f:=x->-x;
```

$$f := x \rightarrow -x \tag{1.2}$$

```
> g:=x->gam*tan(x);
```

$$g := x \rightarrow gam\, \tan(x) \tag{1.3}$$

```
> plot([f,g],0..20,-20..20);
```




```
> eqn:=f(x)=g(x);
```

$$eqn := -x\sim = 2 \tan(x\sim) \tag{1.4}$$

```
> fsolve(eqn,x,1..3);
```

(1.5)

$$2.288929728 \tag{1.5}$$

Problem 4.31

(a)

```
> assume (a, real);
> assume (x,real);
> assume (a>0);
> PsiInitial(x):=sqrt(2/a)*sin(Pi*x/a);
```

$$PsiInitial(x\sim) := \frac{\sqrt{2}\,\sin\left(\frac{\pi x\sim}{a\sim}\right)}{\sqrt{a\sim}} \tag{2.1.1}$$

```
> PsiFinal(x):=sqrt(2/3/a)*sin(Pi*x/3/a);
```

$$PsiFinal(x\sim) := \frac{1}{3}\,\frac{\sqrt{6}\,\sin\left(\frac{1}{3}\,\frac{\pi x\sim}{a\sim}\right)}{\sqrt{a\sim}} \tag{2.1.2}$$

```
> ProbAmp:=int(conjugate(PsiFinal(x))*PsiInitial(x),x=0..a);
```

$$ProbAmp := \frac{9}{8\,\pi} \tag{2.1.3}$$

```
> Probability:=evalf(conjugate(ProbAmp)*ProbAmp);
```

$$Probability := 0.1282346230 \tag{2.1.4}$$

(b)

```
> assume (a, real);
> assume (x,real);
> assume (a>0);
> PsiInitial(x):=sqrt(2/a)*sin(Pi*x/a);
```

$$PsiInitial(x\sim) := \frac{\sqrt{2}\,\sin\left(\frac{\pi x\sim}{a\sim}\right)}{\sqrt{a\sim}} \tag{2.2.1}$$

```
> PsiFinal(x):=sqrt(2/3/a)*sin(2*Pi*x/3/a);
```

$$PsiFinal(x\sim) := \frac{1}{3}\,\frac{\sqrt{6}\,\sin\left(\frac{2}{3}\,\frac{\pi x\sim}{a\sim}\right)}{\sqrt{a\sim}} \tag{2.2.2}$$

```
> ProbAmp:=int(conjugate(PsiFinal(x))*PsiInitial(x),x=0..a);
```

$$ProbAmp := \frac{9}{5\,\pi} \tag{2.2.3}$$

```
> Probability:=evalf(conjugate(ProbAmp)*ProbAmp);
```

$$Probability := 0.3282806349 \tag{2.2.4}$$

(c)

```
> assume (a, real);
> assume (x,real);
> assume (a>0);
> PsiInitial(x):=sqrt(2/a)*sin(2*Pi*x/a);
```

$$PsiInitial(x\sim) := \frac{\sqrt{2}\,\sin\left(\frac{2\,\pi\,x\sim}{a\sim}\right)}{\sqrt{a\sim}} \tag{2.2.1.1}$$

```
> PsiFinal(x):=sqrt(2/3/a)*sin(2*Pi*x/3/a);
```

$$PsiFinal(x\sim) := \frac{1}{3}\,\frac{\sqrt{6}\,\sin\left(\frac{2}{3}\,\frac{\pi\,x\sim}{a\sim}\right)}{\sqrt{a\sim}} \tag{2.2.1.2}$$

```
> ProbAmp:=int(conjugate(PsiFinal(x))*PsiInitial(x),x=0..a)
;
```

$$ProbAmp := -\frac{9}{16\,\pi} \tag{2.2.1.3}$$

```
> Probability:=evalf(conjugate(ProbAmp)*ProbAmp);
```

$$Probability := 0.03205865575 \tag{2.2.1.4}$$