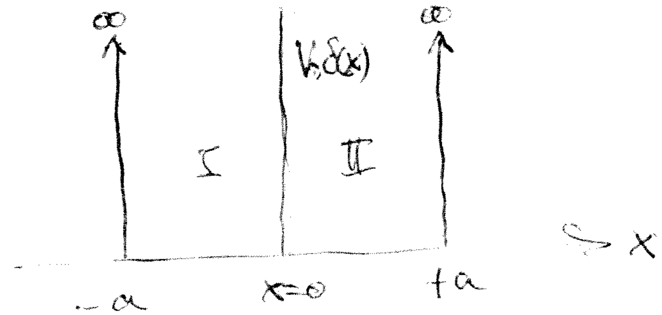
(4.27) A particle of mass m, besides being confined to move in an infinite square well potential of size 2a with walls at $x = \pm a$, is subject to an attractive delta potential

$$V(x) = \begin{cases} V_0 \delta(x) & -a < x < a \\ \infty & \text{elsewhere} \end{cases}$$

where $V_0 > 0$

a) Find the particle's wave function corresponding to even solutions when $E > 0$



Since the Hamiltonian is even, the theorem on page 218 states that non-degenerate eigenstates will be even or odd (have definite parity)

We are asked to study the even solutions.

First, find ψ_I and ψ_{II}:

$$\psi_I = A e^{ikx} + B e^{-ikx} \qquad \psi_{II} = C e^{ikx} + D e^{-ikx}$$

These will work, but are not the most convenient,

The boundary conditions are

$$\psi(\pm a) = 0 \qquad \ldots (1)$$

$$\psi_{II}'(0) - \psi_I'(0) + \frac{2mV_0}{\hbar^2}\psi_I(0) \qquad \ldots (2)$$

$$\psi(0) = \psi_{II}(0) \qquad \ldots (3)$$

A more convenient form for achieving the boundary condition is

$$\psi_I = A \sin k(x+a) + B \cos k(x+a)$$

$$\psi_{II} = C \sin k(x-a) + D \cos k(x-a)$$

Applying the first boundary condition

$$\psi_I(-a) = 0 = B \quad \Rightarrow \quad \psi_I = A \sin k(x+a)$$

$$\psi_{II}(+a) = 0 = D \quad \Rightarrow \quad \psi_{II} = C \sin k(x-a)$$

Appling the third boundary condition, and the desire to find even wave functions

$$\psi_I(0) = \psi_{II}(0) \neq 0 \quad \text{(for the odd states, this must be zero)}$$

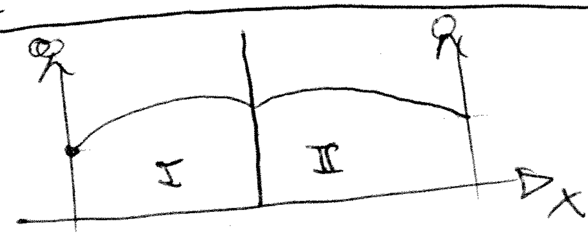
$$A \cdot \sin(+ka) = C \cdot \sin(-ka)$$

$$A = -C$$

So

$$\psi_I = A \sin k(x+a) \qquad k = \sqrt{\frac{2mE}{\hbar^2}}$$

$$\psi_{II} = -A \sin k(x-a)$$



b) Find the energy levels corresponding to even solutions.

From boundary condition 2, we get

$$\left.\begin{array}{l} \psi_{II}' = -A \cos k(x-a) \cdot k \\ \psi_I' = A \cos k(x+a) \cdot k \end{array}\right\} \Rightarrow \boxed{-A \cos ka \cdot k = A \cdot k \cdot \cos ka + \frac{2mV_0}{\hbar^2} A \sin ka}$$

Simplifying

$$-2k\cancel{A}\cos ka = \frac{2mV_0}{\hbar^2}\cancel{A}\sin ka$$

$$k = \frac{2mV_0}{\hbar^2}\frac{\sin(ka)}{\cos(ka)}$$

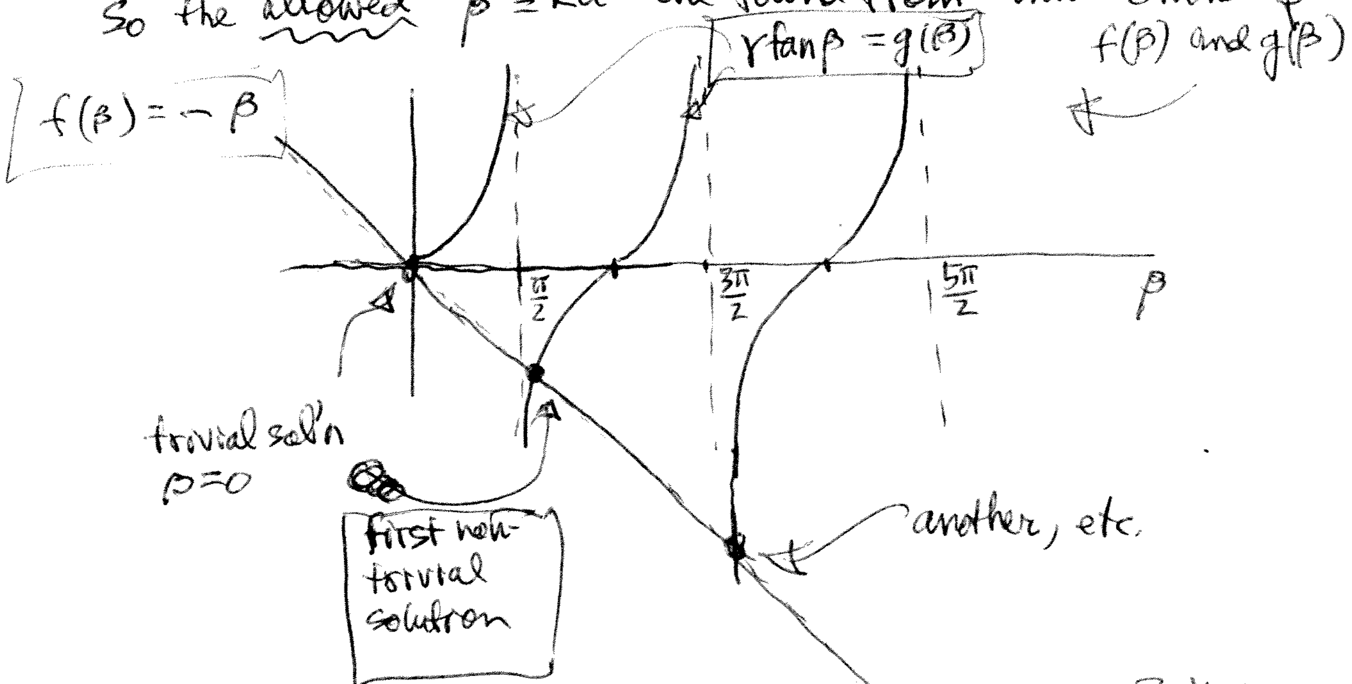
$$-ka = \frac{2mV_0 a}{\hbar^2}\tan(ka)$$

$$f(\beta) \equiv \underbrace{-\beta}_{\text{unitless}} = \underbrace{\boxed{\frac{2mV_0 a}{\hbar^2}}}_{=\gamma}\tan\beta \equiv g(\beta)$$

Is this unitless? $\Rightarrow \left[\frac{2mV_0 a}{\hbar^2}\right] = \left[\frac{kg\cdot \cancel{J}\cdot m\cdot m}{J^{\cancel{2}}\cdot s^2}\right] = \left[\frac{\cancel{kg}\cdot \cancel{m^2}}{\frac{\cancel{kg\,m^2}}{\cancel{s^2}}\cdot \cancel{s^2}}\right] = [1]$ Whew! ☺

$$J = \frac{kg\,m^2}{s^2}$$

So the allowed $\beta \equiv ka$ are found from intersections of $f(\beta)$ and $g(\beta)$



Cannot take solution farther without knowing $\frac{2mV_0 a}{\hbar^2} \equiv \gamma$.